Investment Mathematics and Curated Data Sets

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Objectives

The objectives are

To illustrate common investment mathematics;

To illustrate how equity curves are extended and combined; and

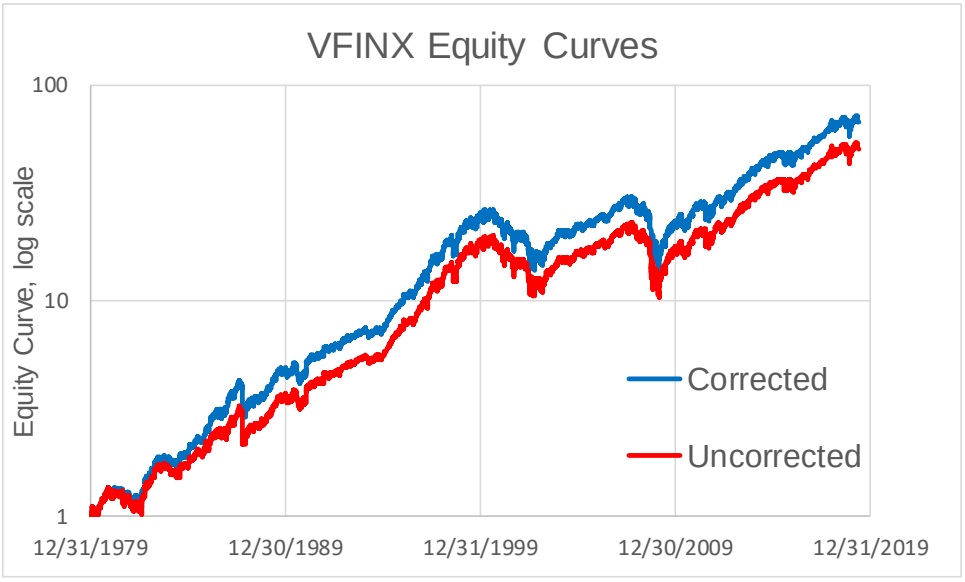
To develop equity curves for the large-cap, mid-cap and small-cap US stock market, for developed foreign stock markets, for real estate, mid-cap US stocks, for cash (intermediate Treasury bonds) and for 4-week T-bills.

Algorithmic definitions can be found at www.lingane.com/qi.

Investment Mathematics

The EQUITY CURVE is the value of an investment portfolio over time. The equity curve is often normalized so that the starting value equals 1.0.

The first chart shows equity curves for VFINX. VFINX is a mutual fund which tracks the S&P 500 Composite® with dividends reinvested.



Source: Large Cap US Stocks.xlsx

The red curve displays the value over time as reported at Yahoo.com while the blue curve shows the corrected values. (The basis for the corrections will be discussed subsequently.)

Equity curves are often plotted on a semi log scale. The vertical "value" axis is logarithmic, and the horizontal "time" axis is linear.

A logarithmic scale allows for a wide dynamic range, which is nearly a hundred-fold in this example. A logarithmic scale means that an incremental vertical displacement represents an incremental percentage difference. If the vertical separation of two equity curves grows or shrinks over time, then one of the equity curves is growing faster than the other.

The vertical displacement of the two equity curves on this chart is nearly constant after 1989, indicating that the primary corrections are prior to 1989.

RETURN is the ratio of the value of the equity curve at one point in time divided by the value at another point in time, minus one. Returns are typically measured over daily or monthly intervals.

The AUGMENTED RETURN is the return plus one. That is, the augmented return is the ratio of the value of the equity curve at one point in time divided by the value at another point in time.

The augmented return is a common mathematical building block. For example, the value of the equity curve at a point in time is the product of the prior augmented returns.

The augmented return is used to splice equity curves.

The annualized return between two points in time, or more formally the compound annual growth rate (CAGR), is computed as the ratio of the value of the equity curve at the more recent date divided by the value of the equity curve at the earlier date, raised to the power 1/T, minus 1. T is the time interval (in years) between the starting and ending dates.

If the ratio of the values of the equity curve is RATIO, CAGR equals

RATIO ^ (1/T) – 1, if T is expressed in years

RATIO ^ (12/T) – 1, if T is expressed in months

RATIO ^ (252/T) – 1, if T is expressed in market days.

The annualized return (CAGR) differs from the average return[1].

SHARPE is a risk-reward ratio in which investment return is divided by investment risk. The numerator is the average EXCESS RETURN and the denominator is the standard deviation of the excess return.

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[1] The annualized return is the fictitious constant annual return which would reproduces the cumulative return over an interval. The average return is the average of the annual returns.

Imagine that the value of a portfolio decreases by 10% in the first year, is unchanged in the second year and increases by 10% in the third year. The average return is zero.

The portfolio value at the end of the three years is 0.9 * 1 * 1.1 = 0.99. The portfolio has suffered a loss, which is not reflected by the average return. The annualized return is a negative 0.3% per year.

EXCESS RETURN is the strategy return minus the return of a risk-free investment such as 4-week T-bills.

If strategy returns are determined monthly, the Sharpe ratio is commonly annualized by multiplying by the square root of twelve. If returns are determined daily, the Sharpe ratio is annualized by multiplying by the square root of 252. These annualizations are approximate.

The larger the Sharpe ratio the better the investment strategy. Annualized values in excess of 1.0 are usually considered very good.

DRAWDOWN is a ratio. The current value of the equity curve is divided by a reference value and the ratio is reduced by 1. Positive values are disregarded.

The reference value is usually the highest prior value. When computing annual drawdowns, the reference value is the highest prior value in the calendar year. MAXIMUM DRAWDOWN over an interval is the largest drawdown measured during that interval.

Drawdowns can be computed from a daily, weekly, monthly or annual equity curve. Drawdowns are largest when computed from a daily equity curve, so be aware of how the values are computed.

ULCER PERFORMANCE INDEX (UPI) is another risk-reward measure. The numerator is the annualized return of the strategy minus the annualized risk-free return[2]. The denominator is the ulcer index, to be defined subsequently.

$$\{ CAGR_{Strategy} - CAGR_{RFR} \} / UI$$

The larger the UPI the better. Values in excess of two are usually considered very good.

Since the values of the Sharpe ratio and of UPI depend on the time interval being investigated, values are less important than the values relative to the value of an appropriate benchmark over the same interval.

The ULCER INDEX is one hundred times the square root of the sum of the squares of the individual drawdowns divided by the total number of intervals, which includes some intervals with zero drawdowns[3]. If drawdowns are expressed as a percentage rather than as a decimal, there is no need for multiplication by one hundred. Drawdown is defined as previously.

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[2] "A popular method for comparing investments with different risk and return is to calculate the excess return (above the risk-free rate) per unit of risk assumed. This is known as investment performance**,** $R_i - R_f$ where R_i is the total return of strategy and R_f is the risk-free rate." Annualized returns are used when calculating investment performance. - Martin and McCann ("The Investor's Guide to Fidelity Funds," second printing Venture Catalyst, Inc. 1992), pp 84 and 86.

[3] Peter G. Martin, who created the Ulcer Index, shows the calculation at www.tangotools.com/ui/ui.htm. Investopedia has a different definition, defining "drawdown" with respect to the maximum value independent of whether the maximum occurred before or after the date at which the drawdown is being measured.

MAR is another risk-reward measure. MAR is the ratio of CAGR to maximum drawdown. As originally defined in the Managed Accounts Report, CAGR and maximum drawdown are measured since the inception of a strategy. The CIMI Group measures MAR over the same intervals as the other performance statistics.

WINS is a measure of the consistency of performance versus a benchmark. WINS is the frequency with which the return exceeds the benchmark return. WINS is typically measured over rolling 36- or 60-month intervals with the 60:40 portfolio as benchmark.

The computation of most of these statistics is illustrated in workbook "Calculating Monthly Stats" of the curated data spreadsheet.

Computing the Dividend-Adjusted Equity Curve

The current value of the equity curve is the product of the prior augmented returns. The augmented daily return is generally the ratio of today's closing price divided by yesterday's closing price.

The augmented return on the date when a dividend is reinvested – the "ex-date" - equals the sum of the closing price on the ex-date plus the dividend divided by prior day's closing price.

Dividend adjusted equity curves must be computed from the current date to earlier dates, not the other way around. Yesterday's adjusted price equals Today's adjusted price divided by Yesterday's augmented return. That is,

$$DAP_{n-1} = DAP_n \div (CP_n + Div_n)/CP_n\text{-}1 = DAP_n * CPn\text{-}1 \;/\; (CP_n + Div_n)$$

where DAP_n and DAP_{n-1} are the dividend adjusted prices today and yesterday, CP_n and CP_{n-1} are the closing prices today and yesterday and Div_n is the dividend.

This formula allows the computation of an equity curve from historical closing prices and historical dividends. The dividend adjusted equity curve must be recomputed whenever there is a new dividend.

Yahoo reports historical closing prices, historical dividends and the dividend-adjusted equity curve. The Yahoo historical prices and dividends do not exactly reproduce Yahoo's dividend-adjusted equity curves, perhaps because Yahoo rounds the reported dividends to three significant figures (± $0.001 per share) while using more precise values to compute the adjustments.

The computation of an equity curve from historical prices and dividends is illustrated in the curated data spreadsheet.

Relative Strength and Its Use

RELATIVE STRENGTH characterizes the relative performance of two equity curves. When the equity curves represent different investment strategies, relative strength characterizes the relative performance of the two strategies.

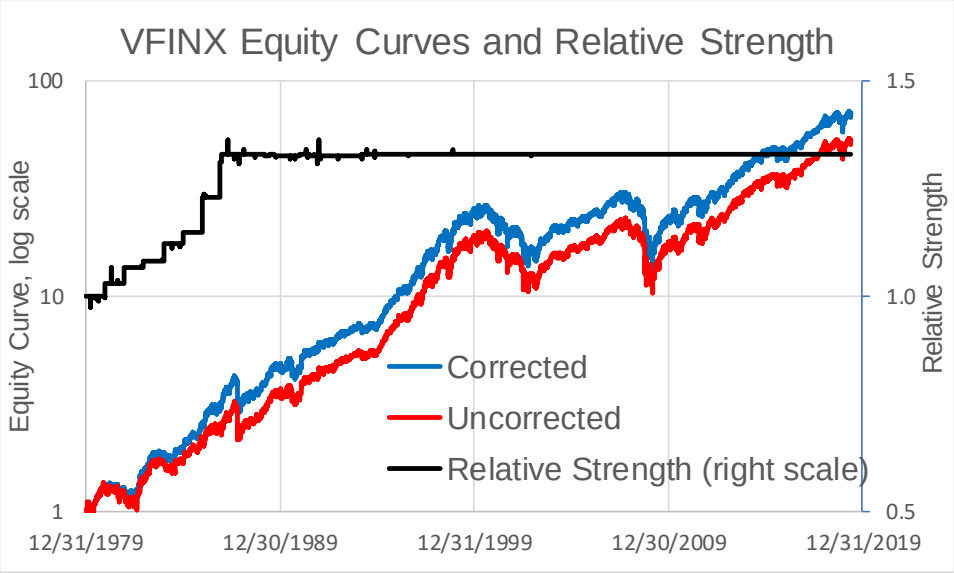
Relative strength is computed as the value of one equity curve divided by the value of another. The relative strength is usually computed over time and normalized so that the initial value is one.

If the relative strength increases over time, the equity curve (strategy) in the numerator is outperforming. If the relative strength declines over time, the equity curve (strategy) in the denominator is outperforming.

The next chart illustrates the relative strength of the corrected Yahoo data for VFINX versus the uncorrected Yahoo data.

The relative strength shows series of steps during the first decade. These steps are the result of adding back the dividends that Yahoo had omitted.

The vertical lines, up or down, reflect the effect of correcting the date on which dividends were paid.



Source: Large Cap US Stocks.xlsx

Relative strength can tease out information which is hidden by annualized returns. For example, the annualized return of the Fidelity Diversified International Fund (ticker FDIVX) was four percentage points per year higher than that of the Vanguard Total International Stock Fund (ticker VGTSX) over the ten years ending December 2007. This large difference in return led many investors to invest in FDIVX. There was so much new money on offer that FDIVX was forced to close to new investors.

A more complete picture of the performance of FDIVX emerges on examining the relative strength over time. As shown in the next chart, FDIVX strongly outperformed VGTSX from the inception of VGTSX in mid-1996 through the end of 2002. The relative strength then began a slow decline which was about equal to the higher expense ratio of FDIVX. In 2005, when assets were pouring into FDIVX, the relative strength showed that the performance of FDIVX was no longer superior.

Relative Strength of Fidelity Diversified International fund (red) versus Vanguard Total International Stock Index fund (green), 7/1996 - 2012. The lowest curve is the relative strength of FDIVX versus VGTSX.

Correcting Yahoo Data

The most common data errors are incorrect dividend values. I have used two techniques to identify and correct dividend errors.

1. iShares publishes historical dividends for iShares ETFs at www.ishares.com. To correct the equity curve of an iShares ETF, recompute the dividend adjusted equity curve using Yahoo closing prices and iShares dividends. Ignore the Yahoo dividend values.
2. Fund companies report annual distributions in the fund's annual reports. Historical reports are accessible from the SEC's EDGAR database.

   The Yahoo dividends should equal the annual distributions reported by the fund company. When the dividend amounts differ, the fund company data are not always granular enough to identify which dividend should be changed, though it is easy when a regular dividend is missing dividend and when dividends that have been entered twice a few days apart.

A second error is missing closing prices. These are repaired by interpolating between adjacent prices and recomputing the equity curve.

Yahoo sometimes does not post dividends exactly on the ex-date. Misplaced dividends can be identified from a plot of the daily return over time. When the daily return spikes up on one day and spikes down by the same amount on the following day – or vice versa - the likely explanation is a misplaced dividend.

Splicing, Blending and Inflation-Adjusting Equity Curves

A SPLICED equity curve occurs when the first part of the data set is based on one equity curve while the later part is based on another. Splicing typically occurs when an index is used to extend an equity curve backwards in time.

For example, the iShares exchange traded fund (ticker EFA) tracks the MSCI-EFA index. The EFA equity curve begins in August 2001. The monthly equity curve can be extended back to the early 1970s by splicing the MACI-EAFE index through August 2001 to the front of the EFA equity curve.

The splicing process begins by calculating augmented returns for both equity curves. Augmented monthly returns are commonly used but augmented daily returns work equally well.

The second step is to splice the augmented returns. The final step is to calculate the spliced equity curve from the spliced augmented returns.

The first two steps are illustrated below for augmented monthly returns. The splice occurs at the end of September 2001.

	MSCI-EAFE Equity Curve	MSCI-EAFE Augmented Return	EFA Equity Curve	EFA Augmented Return	Spliced Augmented Return
12/29/2000	2867.652				
1/31/2001	2866.173	0.9994842			0.9994842
2/28/2001	2651.303	0.9250324			0.9250324
3/30/2001	2474.565	0.9333392			0.9333392
4/30/2001	2646.527	1.0694918			1.0694918
5/31/2001	2553.121	0.9647062			0.9647062
6/29/2001	2448.710	0.9591046			0.9591046
7/31/2001	2404.156	0.9818051			0.9818051
8/31/2001	2343.231	0.9746585	26.7946000		0.9746585
9/28/2001	2105.892	0.8987129	24.2169817	0.9038014	0.9038014
10/31/2001	2159.829	1.0256124	24.6585720	1.0182347	1.0182347
11/30/2001	2239.446	1.0368626	25.4832435	1.0334436	1.0334436
12/31/2001	2252.751	1.0059412	25.6387458	1.0061021	1.0061021

A BLENDED equity curve is the combination of two or more equity curves in a predetermined ratio. For example, the benchmark equity curve is typically a blend of 60% of the equity curve of large cap US stocks plus 40% of the equity curve of investment grade bonds.

In the following example, corrected Yahoo data for VFINX are used to represent US stocks and corrected Yahoo data for IEI are used to represent intermediate bonds.

The process begins by calculating the augmented monthly returns of the equity curves. The second step is to blend the augmented monthly returns in a 60:40 ratio. The final step is to calculate the equity curve from the blended augmented monthly returns[4].

The first two steps in constructing the 60:40 benchmark equity curve from the end of December 2007 are illustrated below.

	VFINX Equity Curve	VFINX Augmented Return	IEI Equity Curve	IEI Augmented Return	Blended Augmented Return	Blended Equity Curve
12/31/2007	107.923		86.536			1.00000
1/31/2008	101.431	0.93985	89.070	1.02928	0.97562	0.97562
2/29/2008	98.133	0.96749	90.453	1.01553	0.98671	0.96265
3/31/2008	97.701	0.99560	91.157	1.00778	1.00047	0.9631
4/30/2008	102.452	1.04863	89.064	0.97704	1.01999	0.98236
5/30/2008	103.768	1.01285	87.971	0.98773	1.00280	0.98511
6/30/2008	95.012	0.91562	88.619	1.00737	0.95232	0.93814
7/31/2008	94.221	0.99167	89.258	1.00721	0.99789	0.93616
8/29/2008	95.584	1.01447	90.132	1.00979	1.01260	0.94795
9/30/2008	87.071	0.91094	91.030	1.00996	0.95055	0.90107
10/31/2008	72.450	0.83208	92.366	1.01468	0.90512	0.81558
11/28/2008	67.252	0.92825	95.921	1.03849	0.97235	0.79303
12/31/2008	67.969	1.01066	97.607	1.01758	1.01343	0.80368

A blended equity curve based on daily returns rebalances daily. A blended curve based on monthly returns rebalances monthly. Daily rebalancing generally results in a higher CAGR than monthly rebalancing.

An INFLATION-ADJUSTED EQUITY CURVE is computed by dividing each augmented nominal return by the augmented inflation return over the same interval. If the return over an interval is 10% and inflation is 3% over the same interval, the augmented inflation-adjusted return is 1.10 / 1.03 = 1.0680.

The inflation-adjusted equity curve is computed from the augmented inflation-adjusted returns.

[4] Calculating the blended equity curve from monthly returns is equivalent to monthly rebalancing. One could also calculate the blended equity curve using daily returns; this would be equivalent to daily rebalancing.

Computing Monthly Statistics from a Daily Equity Curve.

Statistics computed from a daily equity curve are generally different from statistics computed from a monthly equity curve. The exception is CAGR. It would be useful, therefore, for CIMI to standardize on a uniform presentation method. My preference is to derive statistics from a monthly equity curve since it is not possible to compute daily equity curves prior to about 1990.

To compute monthly statistics from a daily equity curve, the first step is to extract the month-end values of the equity curve. I find it convenient to use the VLOOKUP function in EXCEL, as is illustrated in workbook "Monthly Stats" of the curated data spreadsheet.

Performance statistics are then computed from the month-end values in the usual fashion.

Monthly performance statistics generally appear to be better than daily statistics, as shown below. The example is a timed portfolio of large cap US stocks from December 31, 1997 through May 31, 2019, a total of 257 months. The daily statistics were provided by Don Maurer.

	Monthly Stats	Daily Stats
CAGR	10.5%	10.5%
Sharpe	0.875	0.71
MaxDD	0.153	0.190
MAR	0.69	0.55
Ulcer Performance Index	2.21	2.09

Annualizing Monthly Standard Deviations

The value of the standard deviation depends on the data interval. Daily returns produce a daily standard deviation whereas monthly returns produce a monthly standard deviation. How then does one convert a daily or monthly standard deviation to an annual basis?

The quick and dirty approach is to approximate the annual standard deviation as the square root of twelve times the monthly standard deviation, or as the square root of 252 times the daily standard deviation.

A better approach is to apply the formula[5]

$$\sigma_n = \text{SQRT}\ \{\ [\ \sigma^2 + (\ 1 + \mu\ )^2\ ]^n - (\ 1 + \mu\ )^{2n}\ \}\ \text{where,}$$

n = number of periods per year; 12 for monthly data.

σ = standard deviation for the period.

μ = arithmetic mean for the period.

[5] Ibbotson's "Stocks, Bonds, Bills and Inflation Yearbook" published by Duff & Phelps LLC attributes this formula to Levy and Gunthorpe, *Journal of Portfolio Management*, Summer 1993.

Query. Would be Sharpe ratio be the same, when computed from monthly and daily values, if the Levy and Gunthorpe equation were used to scale monthly standard deviation to an annual value?

Monte Carlo Simulations.

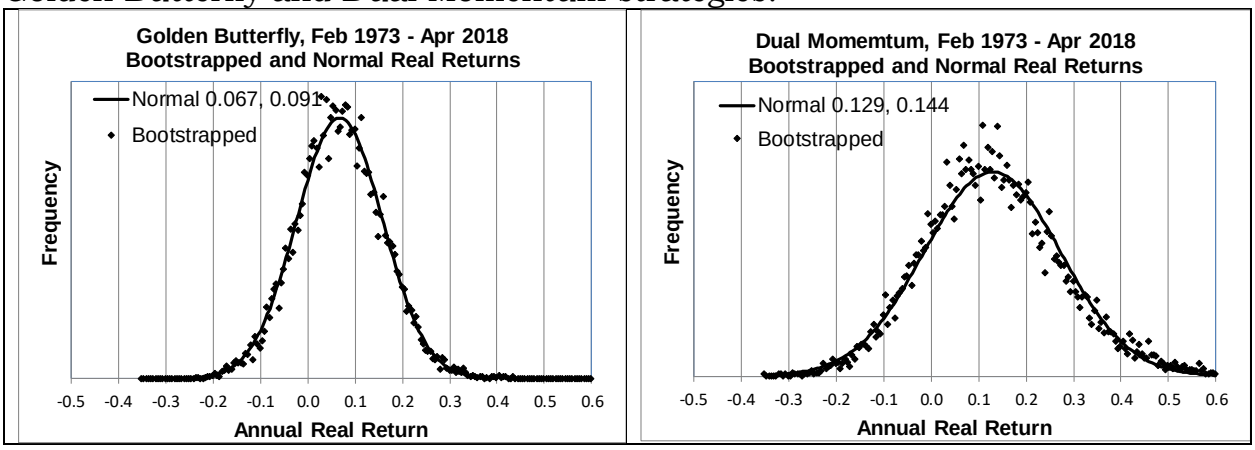
Monte Carlo simulations are useful for estimating the longevity of an investment strategy after retirement. Monte Carlo simulations are easily conducted at PortfolioVisualizer.com. The required inputs are the distribution of investment returns, the distribution of inflation returns and the correlation between inflation and investment returns.

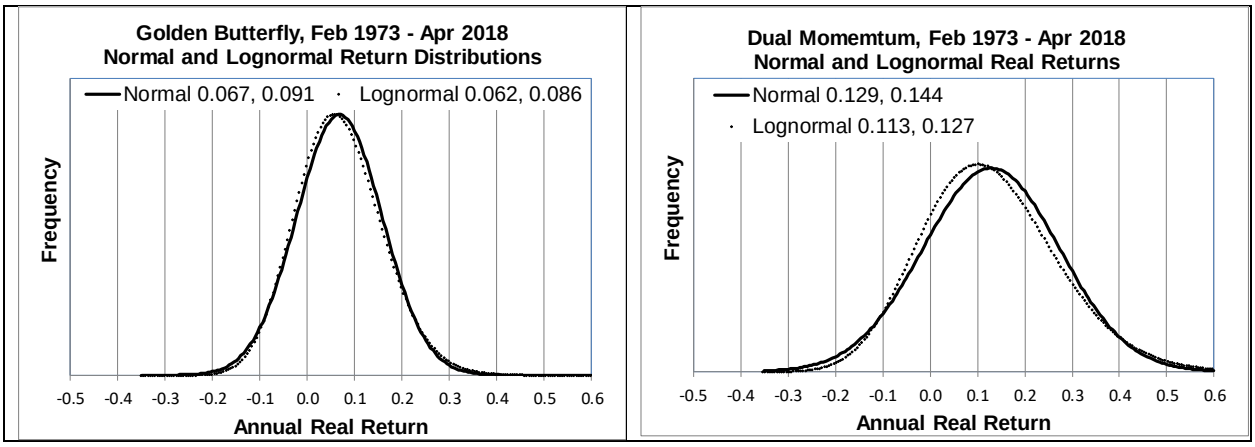
My preference is toy computing the distribution of annual inflation-adjusted investment returns. This has the advantage of eliminating the correlation parameter.

The central limit theorem says that the distribution created from the product of random variables tends towards the lognormal distribution. Since annual real returns are typically the product of twelve, monthly nominal returns each of which has been divided by the inflation return, it is not surprising that the return distributions of annual real investment returns are approximately lognormal.

Normal and lognormal distributions are almost identical for the values of standard deviation typical of investment returns. Hence the real return distributions also approximate the normal distribution.

The next chart compares the normal and lognormal distributions for the Golden Butterfly and Dual Momentum strategies.





Source: Retirement Risk.xlsb

"Bootstrapped" data are the result of fitting a histogram to the simulated returns using EXCEL's HISTOGRAM function in the Analysis ToolPak.

Normal distributions are calculated using EXCEL's NORMDIST function using the mean and standard deviation of the simulated returns as input parameters.

Lognormal distributions are computed using EXCEL's LOGNORMDIST function. The direct approach to the parameters of the lognormal distribution is to compute the mean and standard deviation of the logarithms of the simulated returns.

The easier approach is to calculate the lognormal parameters from the arithmetic mean μ and standard deviation σ of the simulated returns. The mean of the logarithms of the returns is

$$\mathrm{LN}\left((1+\mu) \,/\, \mathrm{SQRT}\left(1 + (\sigma/(1+\mu))^2 \right) \right).$$

The standard deviation of the logarithms of the returns is

$$\mathrm{SQRT}\left(\mathrm{LN}\left(1 + (\sigma/(1+\mu))^2 \right) \right).$$

Curated Data Sets

Standardization of the historical price data improves comparisons among simulated strategies. The goal is to construct reduced error, "curated" daily equity curves (from which daily returns can be calculated) for

- Large cap US stocks;
- Developed foreign stock markets;
- US real estate stocks;
- Mid-cap US stocks;
- Small-cap US stocks;
- Intermediate Treasury bonds; and
- 4-week (often referred to as "1-month") Treasury bills.

These data sets are important because several allocation and market timing strategies have been developed based on back testing with these assets. T-bill returns are used in some timing algorithms and to represent the risk-free rate for purposes of the Sharpe ratio and the Ulcer Performance Index.

Curated equity curves were developed from December 31, 1997. These equity curves provide daily coverage of the two most recent bear/bull market cycles[6].

The daily equity curves are reasonable extensions of the monthly index data used in prior studies.

- SBBI large cap US stock returns[7];
- MSCI-EAFE index of developed foreign markets;
- FTSE NAREIT index of US REITS;
- No prior mid cap index;
- SBBI small cap US stock returns;
- SBBI intermediate government bond returns; and
- SBBI 1-month Treasury bill returns.

Equity curves calculated from the extension data are compared to the equity curves calculated from the index data in the overlap interval.

The recommended equity curves are available in the curated data spreadsheet.

Large Cap US Stocks. SBBI provides monthly return data for large cap US stocks from January 1926. Extensions were evaluated with respect to this index.

Yahoo provides daily data from 1980 for the Vanguard Index 500 fund (ticker VFINX) which tracks the S&P 500 Composite with dividends reinvested[8]. There are numerous errors, especially in the earlier years.

Comparing the daily returns of Y_VFINX to the daily returns of the S&P Composite without dividends (Y_^GSPX) allowed the correction of misplaced or understated dividends and of errors in the closing prices.

After correction, the Yahoo data show good relative strength behavior versus the SBBI data, as is shown in the next chart.

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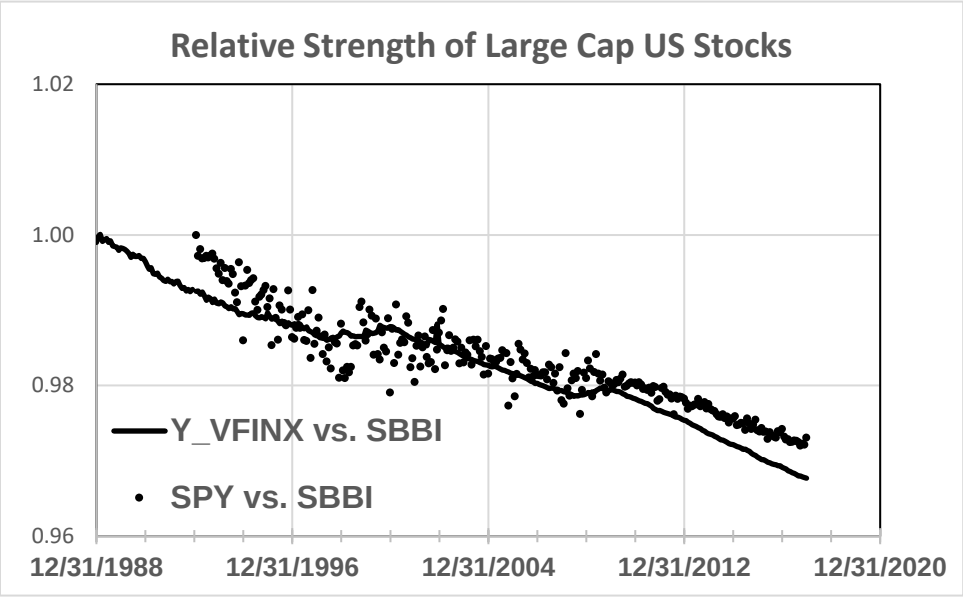
[6] Earl Adamy ("Forty Shades of Tactical Asset Allocation Across Bull and Bear Markets", January 17, 2018 at seekingalpha.com) says that the two most recent market cycles are May 2000 – August 2007 and September 2007 to the present. Since some algorithms require twelve months to initialize, data sets should start before May 1999.

[7] SBBI refers to Ibbotson's "Stocks, Bonds, Bills and Inflation Yearbook" published annually by Duff & Phelps LLC.

[8] The prefix Y_ is added to distinguish data sourced from Yahoo.com. All Yahoo data presented herein have been corrected for obvious errors.

(Relative strength declines over time because index data are free of expenses.)



Source: Data&Timers.xlsb

SPY tracks the S&P Composite, with dividends reinvested, and there are data from 1993. As shown in the chart, the relative strength trend of SPY versus SBBI is similar to the trend with Y_VFINX but the SPY trend exhibits more short-term variations than the Y_VFINX data prior to about 2010.

SPY could be used as a simulation vehicle after about 2010, but it is simpler to stay with the corrected Y_VFINX data.

The quality of the YAHOO SPY data was not tested.

Recommendation. SBBI monthly returns through 1997 and corrected Y_VFINX daily returns thereafter.

Foreign Developed Markets. The MSCI-EAFE (Europe, Australasia, Far East) Index is designed to measure the performance of developed equity markets, staexcluding the US & Canada. This index consists of twenty-one developed markets: Australia, Austria, Belgium, Denmark, Finland, France, Germany, Hong Kong, Ireland, Israel, Italy, Japan, the Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, and the United Kingdom.

Monthly data are available from 1970 at msci.com.

The index used here is denominated in US dollars and is net of foreign tax since the returns of US mutual funds and ETFs are denominated in dollars and are net of foreign tax. (Foreign tax paid can be recovered on a US tax return in some circumstances.)

MSCI_EAFE is the base foreign stock index and daily extensions are evaluated with respect to this index.

I have used Harbor International (HAINX), an actively managed fund with an enviable record, as my foreign stock surrogate in the past. However, Harbor International is not a surrogate for the MSCI-EAFE index because the relative strength has doubled over the past twenty years.

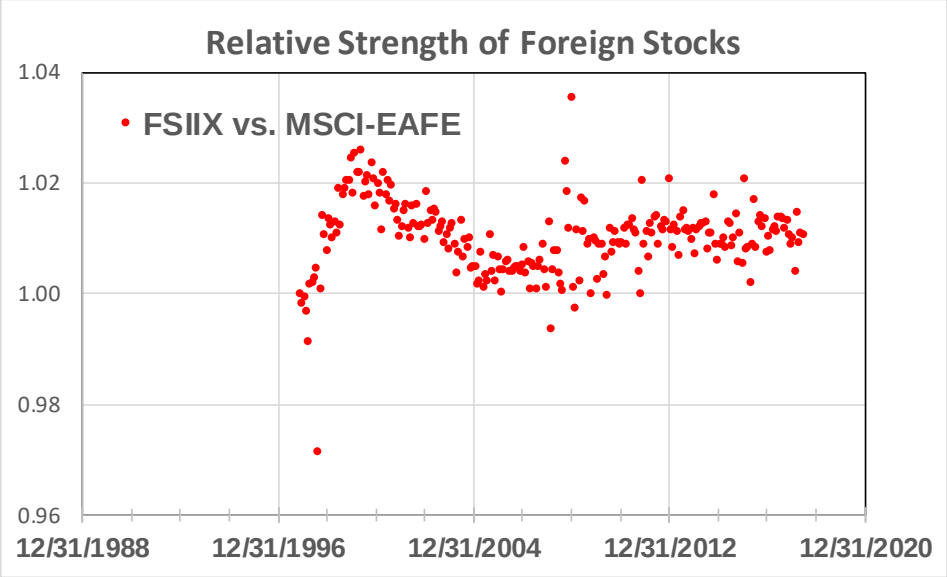
Fidelity introduced its International Index Fund (ticker FSIIX) in November 1997. The equity curve was calculated from Yahoo closing prices and the dividends reported by Fidelity.

As shown in the next chart, the relative strength of Y_FSIIX vs MSCI-EAFE has varied between approximately 1.00 and 1.02.

It is surprising that the relative strength is stable because a mutual fund has expenses whereas an index does not. Apparently, the sampling techniques and/or currency hedging that Fidelity uses in tracking the MSCI-EAFE index provide enough upside to cover the fund's expenses.

Y_FSIIX is a reasonable surrogate for the MSCI-EAFE index because the relative strength does not exhibit wide variations. The history is longer than that of most other possibilities. (See subsequent comments about SWISX.)

Fidelity merged this share class into the institutional class (FSPSX) of the same fund in November 2018. We need, therefore, an extension for FSIIX. Possibilities include VEU, EFA and FSPSX.



Source: Data&Timers.xlsb

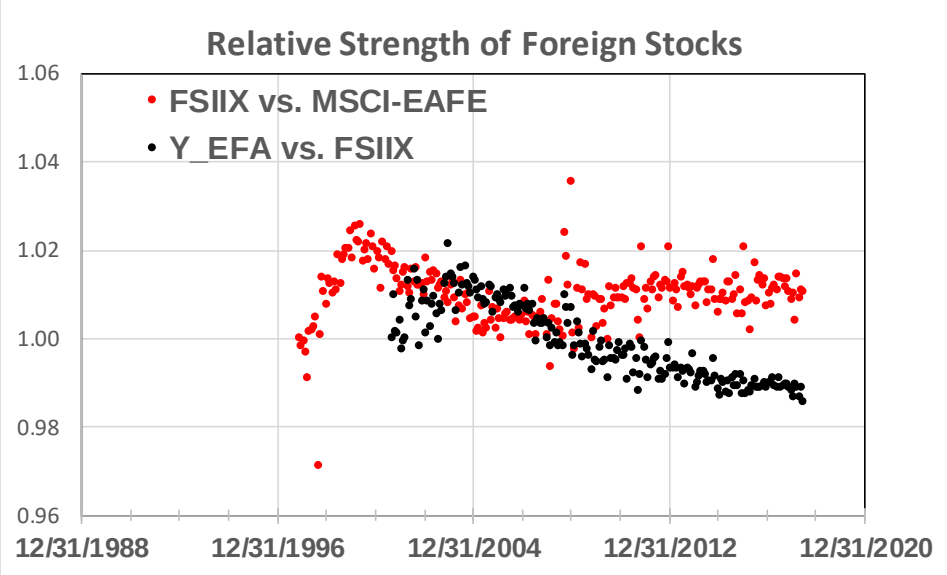
VEU is the ETF share class of Vanguard's FTSE All-World ex-US Index Fund. VEU was introduced in March 2007. The fund initially tracked the MSCI-EAFE index, but it has tracked the FTSE Developed ex-North America index since 2012.

The relative strength of VEU vs. MSCI-EAFE shows a variation from 1.0 to 1.4 and back to about 1.06 currently. While VEU could serves as a vehicle for

trading foreign stocks, VEU is an unsatisfactory surrogate for the MSCI-EAFE index.

iShares introduced an exchange traded fund tracking the MSCI_EAFE index (ticker EFA) in August 2001. The equity curve for Y_EFA was recomputed using the closing prices reported by Yahoo and the dividends reported by iShares[9].

Y_EFA has steadily underperformed Y_FSIIX by about 15 bpts annually; see the black dots in the following chart. The degree of underperformance is about equal to the difference in expense ratios.

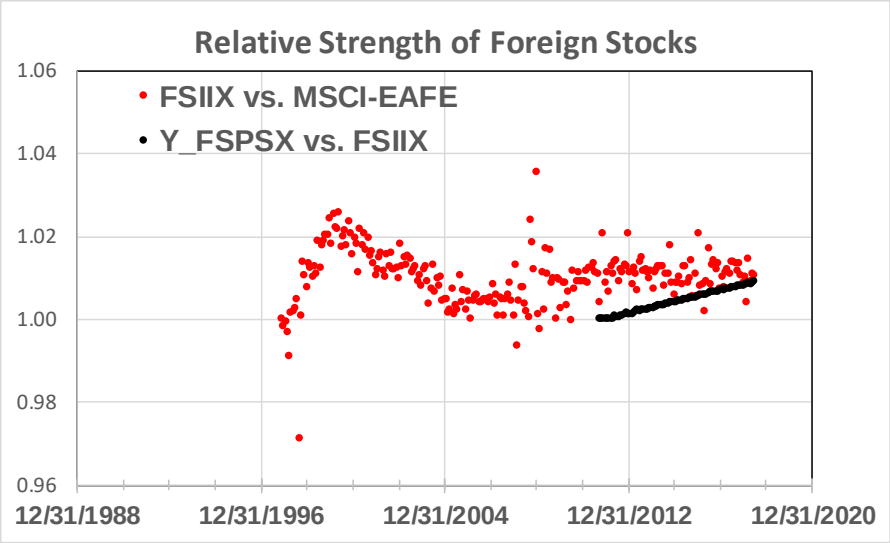


Source: Data&Timers.xlsb

Dividend adjusted prices for Y_FSPSX were recomputed because Yahoo claimed the December 2014 dividend on both December 19 and again on December 22 and because the Yahoo reported no closing price for May 19, 2016.

The relative strength of Y_FSPSX vs. Y_FSIIX exhibits low volatility and a positive slope of about 10 bpts annually. The low volatility confirms that the portfolios are the same. The positive slope likely reflects FSPSX's lower expense ratio.

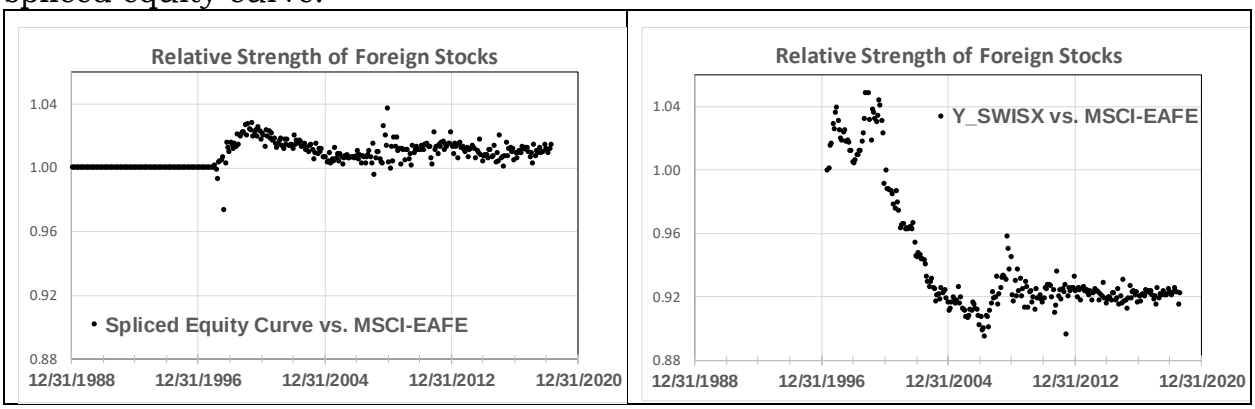
[9] www.ishares.com/us/products/239623/ishares-msci-eafe-etf#distributionsDialog.



Source: Data&Timers.xlsb

Schwab's SWISX is a potential alternative to the spliced equity curve of FSIIX and FSPSX. There is history from 1997. SWISX adopted the MSCI-EAFE benchmark in late 2011, having used a proprietary benchmark previously.

A plot of the relative strength of the uncorrected Yahoo equity curve vs. the MSCI-EAFE index (below, right) shows a wider variation than the relative strength of the spliced equity curve (below, left). The relative strength plot using FastTrack data is similar. The conclusion is that SWISX is inferior to the spliced equity curve.



Recommendation. MSCI-EAFE monthly returns through 1997, corrected Y_FSIIX daily returns through 2012 and corrected Y_FSPSX daily returns thereafter.

Real Estate. The FTSE NAREIT All REITs Index is a capitalization-weighted index that includes all tax-qualified real estate investment trusts (REITs) listed on the New York Stock Exchange, on the American Stock Exchange or on the NASDAQ National Market List. Monthly data from 1972 are available at nareit.com.

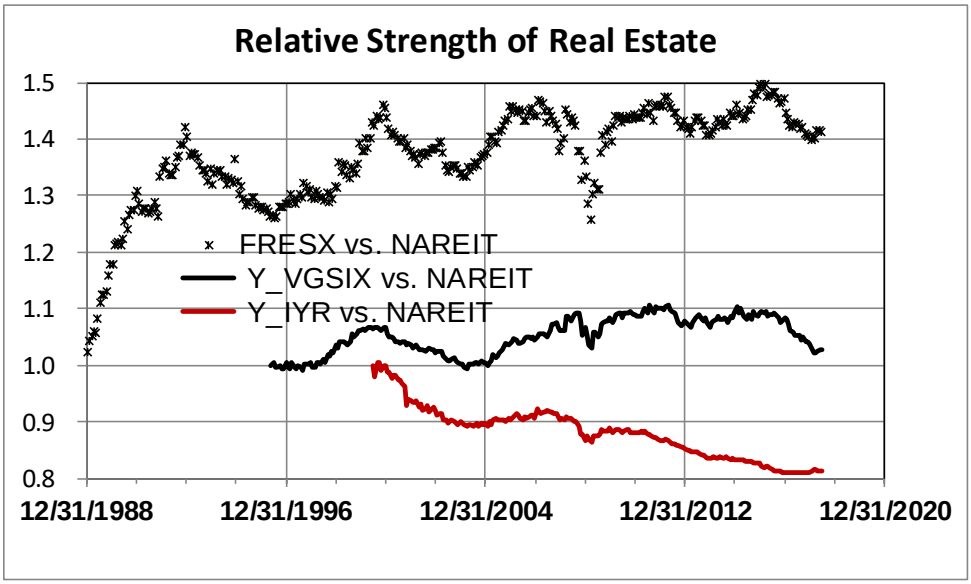
The Vanguard REIT index fund (share classes VGSIX and VNQ) tracks a different index, the MSCI US REIT index. There are mutual fund data from

May 1996 and ETF data from September 2004. The relative strength varies between 1.0 and 1.1 as can be seen in the next chart.

The dividend adjusted prices for Y_VGSIX were recomputed since Yahoo claimed the December 2014 dividend on both December 19 and again on December 22. At my suggestion, FastTrack confirmed the entire dividend history of VGSIX with Vanguard.

The iShares Dow Jones US Real Estate fund (IYR) tracks the Dow Jones US Real Estate Index. There are data from June 2000.

The bottom curve in the chart shows the relative strength of Y_IRY vs. the NAREIT index. Y_IRY is unsatisfactory because it steadily underperforms by a hundred basis points annually. (The underperformance is probably not tracking error but the fact that IRY tracks the Dow Jones US Real Estate index.)



Source: Data&Timers.xlsb

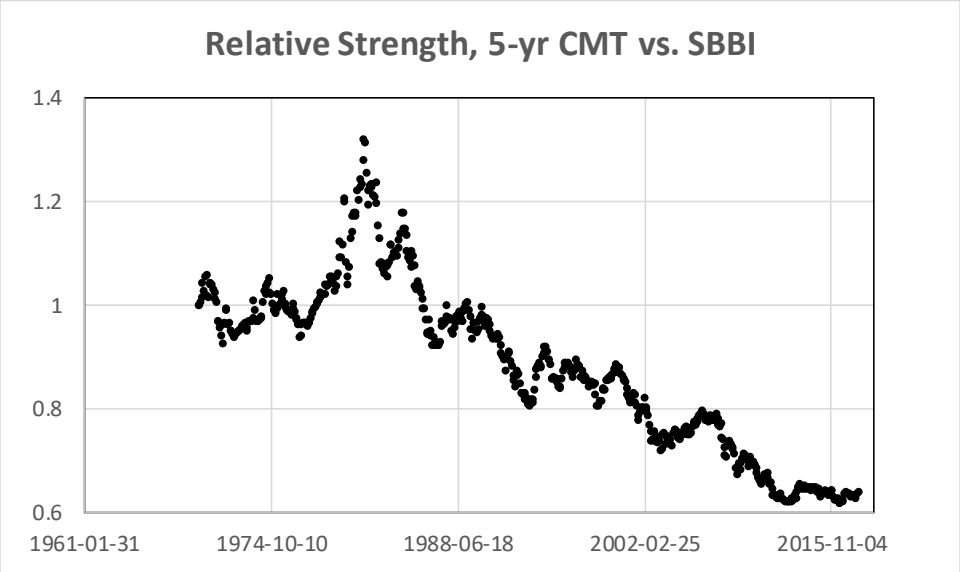
Fidelity Real Estate (FRESX) is an actively managed fund which invests in US and foreign real estate operating companies. FRESX had been my real estate simulation surrogate.

FRESX outperforms both VGSIX and NAREIT by about one hundred basis points annually. While outperformance is great for a portfolio constituent, it is a disqualifier as an index surrogate.

Recommendation. NAREIT monthly returns through 1997 and the corrected Yahoo daily returns for VGSIX thereafter.

CASH. SBBI intermediate term government bond returns were determined as the return of a 5-year Treasury bond over the following year.

The Treasury publishes daily the returns of a hypothetical Treasury bond with a constant 5-year maturity. The plot of relative strength indicates that these returns are different from the SBBI data.

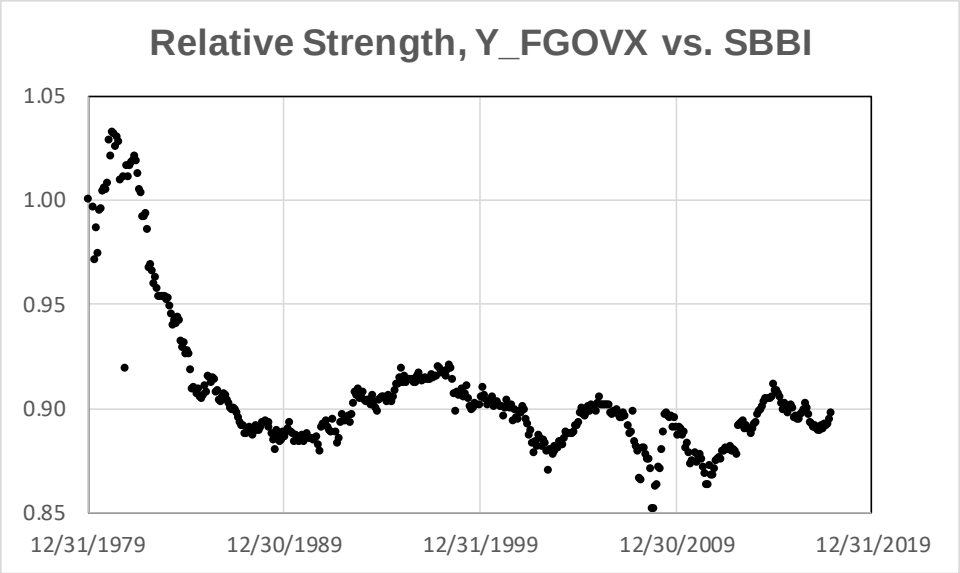


Source: constant maturity treasury.xlsx

The relative strength comparing the 7-year constant maturity Treasury index to SBBI is equally disappointing.

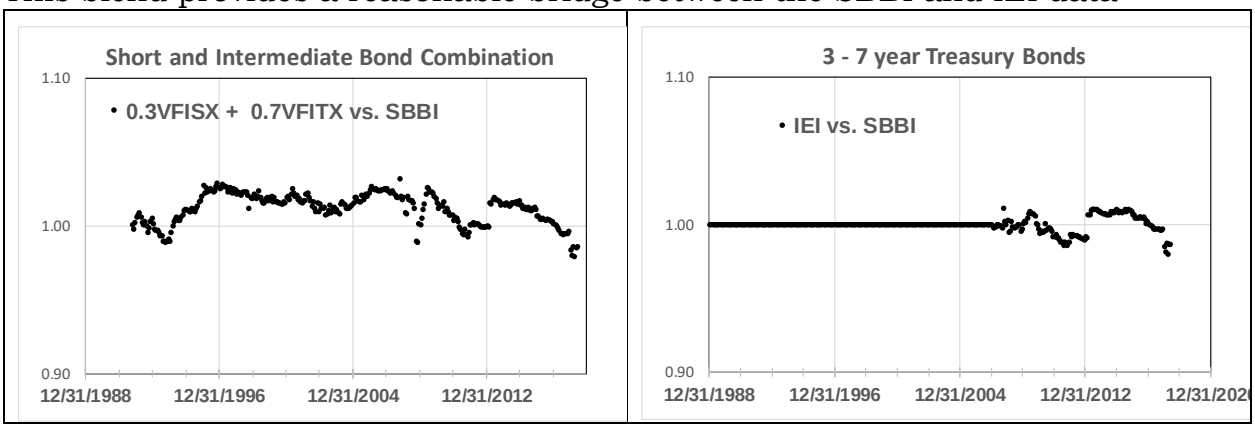
Yahoo reports daily data for FGOVX, the Fidelity Government Income fund, from 1985 and monthly data 1980 - 1984. Several obvious instances where Yahoo posted dividends on the wrong date were not corrected. Adding the April 30, 2010 dividend, which Yahoo omitted, brought the 2010 annual return for Y_FGOVX into registration with the annual return reported by Fidelity.

The Fidelity Government Income fund is an actively managed fund that has morphed over the years from a GNMA fund to a fund containing half Treasuries and half collateralized market obligations. FGOVX is not a suitable surrogate for CASH because the relative strength vs. SBBI is variable, especially before 1990. See the next chart.



IEI (a 3-7-year Treasury ETF) provides a stable relative strength versus SBBI. The equity curve for IEI was calculated from the Yahoo closing prices and the distributions reported by iShares.

IEI has no data before about 2008, but a blend of 30% VFISX (a 1-5-year Treasury bond mutual fund) and 70% VFITX (a 5-10 years Treasury bond mutual fund) was found to exhibit a stable relative strength after about 1992. This blend provides a reasonable bridge between the SBBI and IEI data



Source: Data&Timers.xlsb

Recommendation. Represent "CASH" by SBBI intermediate term government bond monthly returns prior to 1997, by a 30:70 daily blend of VFISX and VFITX prior to 2008 and by corrected daily Y_IEI returns thereafter.

Risk Free Rate. SBBI provides 1-month T-bill returns from 1926 through 2017. The Ken French data library publishes the historical SBBI returns and extends the data to within a month or two of the present.

Treasury and the Federal Reserve publish the "constant maturity" yield for the 1-month T-bill. Federal Reserve data tend to have a one-day latency.

Yields for 1-month T-bills are equivalent to annualized returns. To generate a daily equity curve from yield data, the daily return is estimated from the yield using the following formula.

$$\text{Daily Return} = (1 + \text{yield}/100)^{\wedge}(1/252)$$

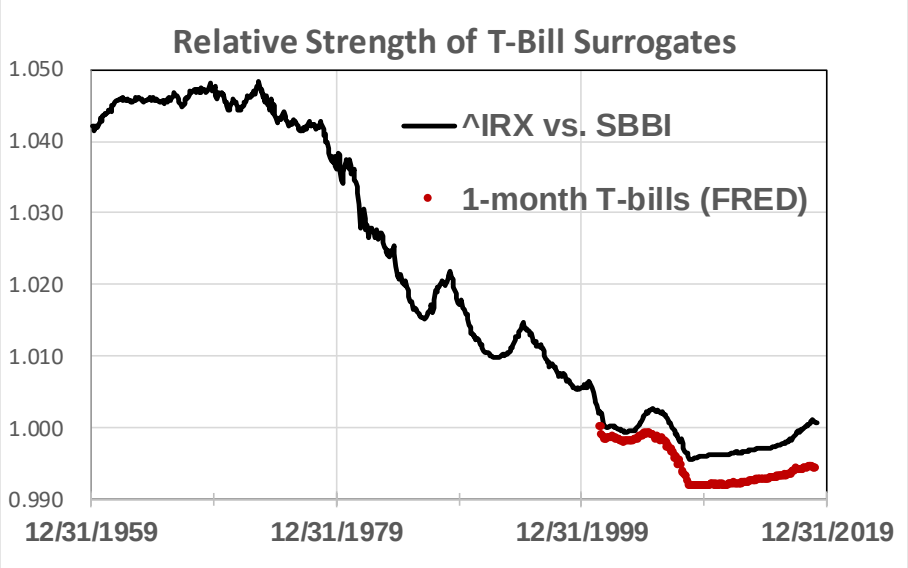
The factor of one hundred converts the yield data from a percentage format to a decimal format. The daily equity curve is calculated from the daily returns.

The constant maturity 1-month T-bill yields approximate the SBBI returns because the relative strength curve is bounded between 0.99 and 1.00.

Daily constant maturity T-bill yield data are only available from July 31, 2001. We need a bridge method to provide earlier data.

The CBOE 13-week T-bill index has daily data from 1960. Index values are available from Yahoo with the symbol ^IRX. Treasury publishes "constant maturity" daily investment equivalent yields for the 13-week T-bill from January 1990. The results should be comparable to the CBOE index, but this was not tested.

The systematic change in relative strength over the past sixty years, shown in the next chart, demonstrates that the CBOE 13-week T-bill index is not an attractive surrogate for the SBBI risk free rate. Nonetheless, we have used the CPOE 13-week index to bridge the 4-year interval from 1998 through 2001.



Source: Data&Timers.xlsb

Recommendation. Represent the risk-free rate by monthly French data library risk-free returns through 1997, by Y_^IRX daily yields through 2001 and by daily 1-month constant maturity Treasury yields thereafter.

Mid Cap US Stocks (S&P 400)

The Dreyfus Midcap Index Fund (ticker PESPX) seeks to track the S&P 400 mid cap index. The equity curve begins in June 1991, which makes this one of the

longest equity curves in the mid-cap space. Yahoo data were corrected by assuming that the closing prices are correct and substituting the dividends from the Dreyfus Annual Reports (EDGAR). There are differences in the dividend amounts prior to 2005.

The equity curve for IJH begins in May 2000; this is the longest mid-cap ETF available. The Yahoo data were corrected by assuming that the closing prices are correct and substituting the dividends from the iShares website.

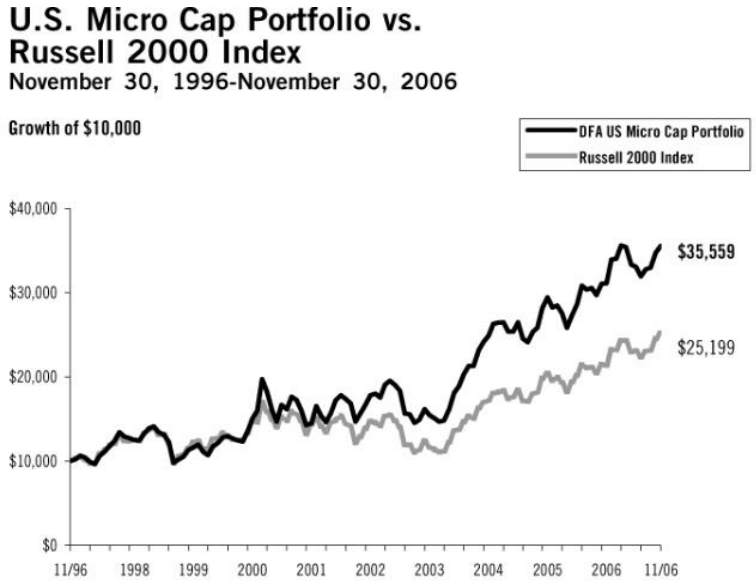
Yahoo has data for the S&P 400 Mid Cap index from 1981, ticker ^MID. The index values are not dividend adjusted.

Recommendation. Represent the equity curve of dividend adjusted mid-cap US stocks by the corrected Yahoo data for PESPX.

Small Cap US Stocks

When we think of small cap stocks, we usually think of the Russell 2000 index. The SBBI small cap index represents smaller stocks than the Russell 2000[10].

The following chart is taken from the DFA annual report for the year ending November 30, 2006.



The next chart shows that the divergence of the two indices in the runup to the 2002-4 bear market. "Lo10", "Lo20" and "Lo30" are the smallest 10, 20 and 30% fractions in the Ken French database.

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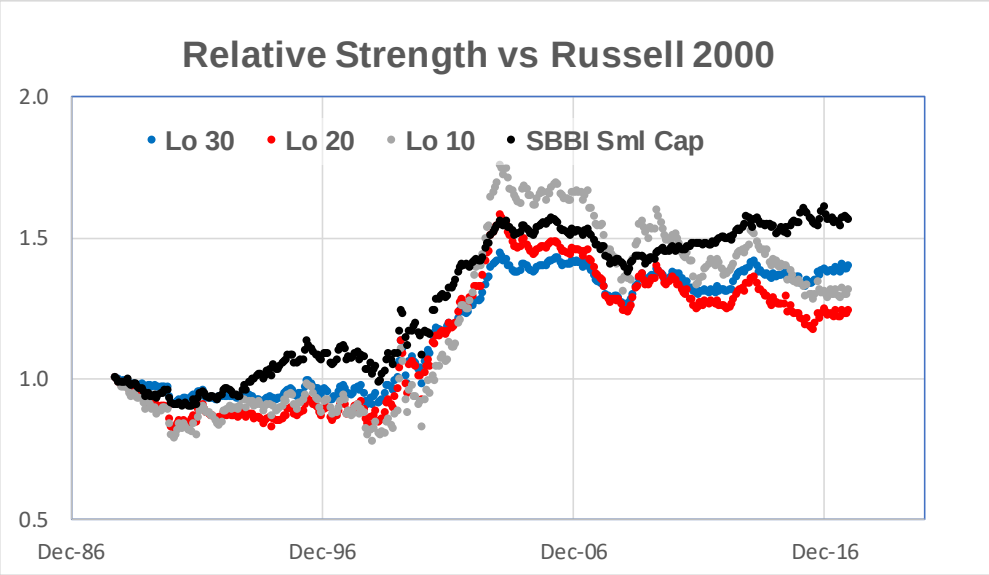
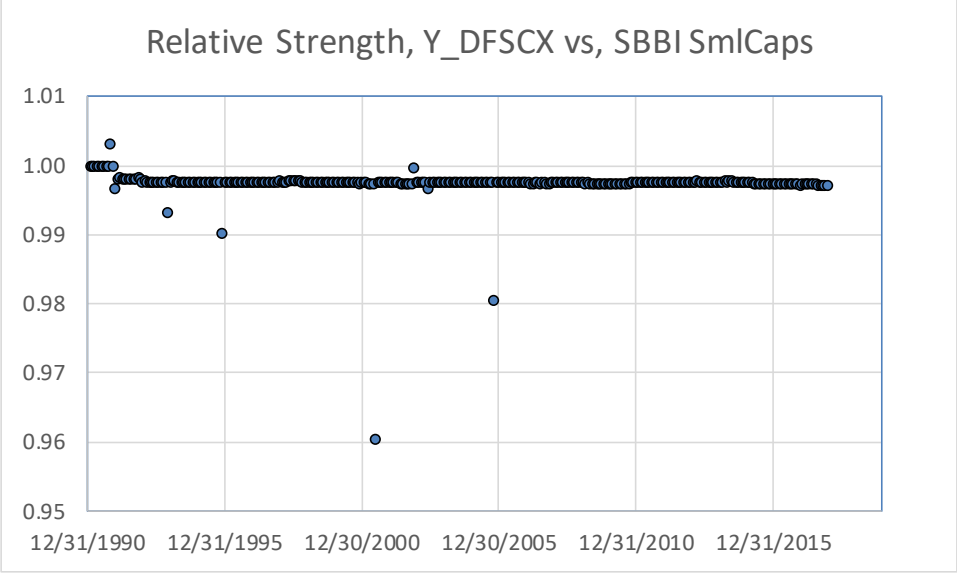
[10] The Russell 2000 represents the bottom two thirds of US stocks by market cap whereas the SBBI Small Cap represents approximately the bottom twenty percent.

SBBI represented the returns of the bottom quintile (by market capitalization) of the New York Stock Exchange through 1981. Beginning in 1982, the returns are those of the Dimensional Fund Advisors (DFA) Small Company 9/10 Fund, ticker DFSCX. This fund targets the smallest 20%, by market cap, of US listed companies. In 2001, DFA changed the name of the fund to the DFA Micro Cap Fund (same ticker) and modified the strategy to track the smallest 5%, by market cap, of US listed companies.



If you wish to track the SBBI index, the daily extension is based on DFSCX. The historical daily equity curve for DFSCX was calculated from 1986 by adjusting the Yahoo dividends to match the dividends in DFA annual reports or Morningstar.

The chart of relative strength shows a half dozen outliers, probably due to errors in closing prices. For example, the 0.96 relative strength in June 2001 is due to an error in the June month-end closing price.



The small downshift in relative strength in December 1991 disappears if the dividend is increased by one cent, which is about the precision of the reported dividends.

If you prefer to track the Russell 2000, Vanguard offers the small cap mutual funds, NAESX, which has daily data from 1985. NAESX was benchmarked

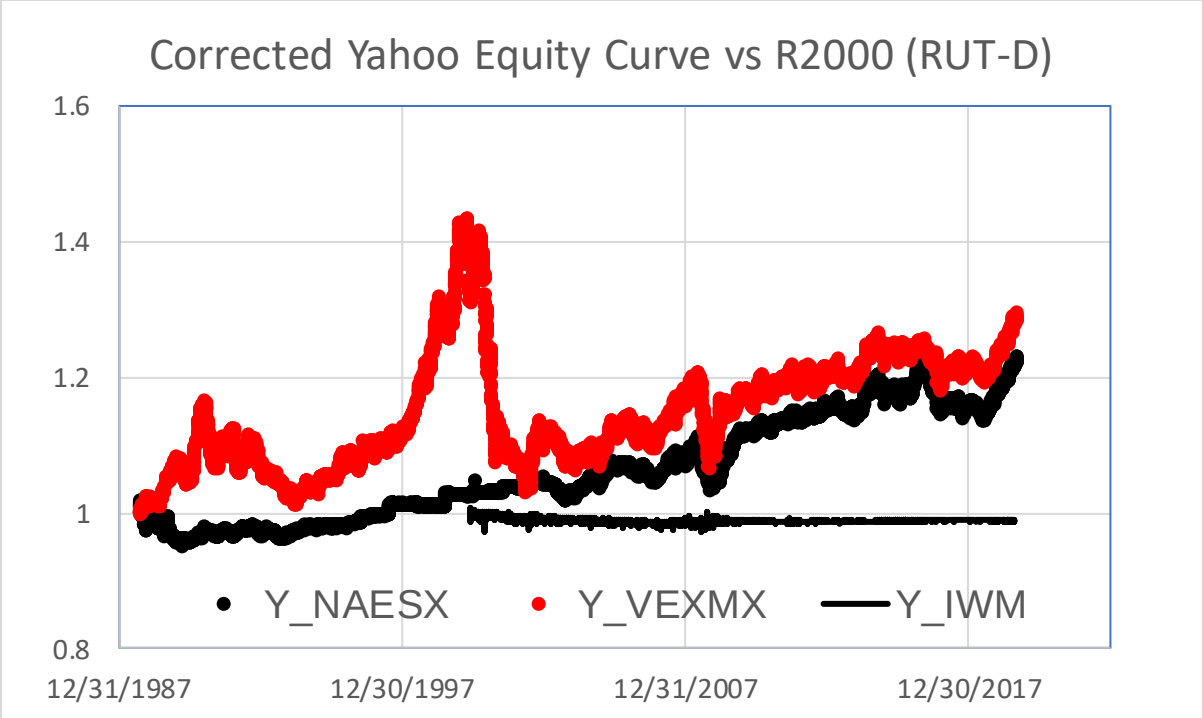
against the Russell 2000 Index through May 2003, the MSCI US Small Cap 1750 Index through January 2013 and the CRSP US Small Cap Index thereafter.

Equity curves were recomputed after adjusting Yahoo closing prices in October 1987 to match the daily returns of the Russell 2000 index (^RUT at Yahoo) and after adjusting dividends to match the dividends reported in Vanguard annual reports or by Morningstar.

Vanguard offers a second small cap fund, VEXMX, with daily data from 1988. This fund was benchmarked against the DJ Wilshire 4500 Index through June 2005 and against the SP Completion Index thereafter.

iShares offers IWM, an exchange traded fund which tracks the Russell 2000 Index and daily history from May 2000. The IWM equity cure was recomputed after replacing the Yahoo dividends by the dividends from the iShares website.

The relative strengths of the corrected equity curves for NAESX, VEXMX and IWX are shown vs. the Russell 2000 index with dividends in the following chart. WEXMX is a poor match.



Recommendations.

Use SBBI small cap month-end data through 1986 and corrected Yahoo daily data for DFSCX after 1986.

Alternatively, discard the SBBI small cap index in favor of the Russell 2000. This means using the corrected Yahoo daily data for NAESX from 1985 through 2000 and the corrected IWM data thereafter.

S&P 600 Small Cap Index

There is a third small cap index. The Vanguard ETF (ticker VIOO with history from September 2000) and the Principal SmallCap S&P600 Index fund (ticker PSSIX with history from December 2000) track the S&P600 index.

The substantial return divergence between the S&P SmallCap 600 and the Russell 2000 is a widely documented and researched investment topic. Most of the S&P SmallCap 600's excess returns have stemmed from differences in index construction, such as the profitability, liquidity, and public float criteria that are absent in the Russell 2000. Fundamental and macroeconomic risk factor attribution analyses show that the S&P SmallCap 600 has had stronger bias to the quality factor and has performed better during market environments in which higher quality has outperformed lower quality.

We show that benchmark selection matters when it comes to distinguishing a successful manager from an unsuccessful one. The S&P SmallCap 600 has been the more difficult small-cap benchmark to beat of the two, with a higher percentage of actively managed small-cap funds underperforming it[11].

[11] A Tale of Two Small-Cap Benchmarks: 10 Years Later, Phillip Brzenk, Bill Hao and Aye M. Soe, S&P Dow Jones Indices, September 2019. https://us.spindices.com/documents/research/research-a-tale-of-two-small-cap-benchmarks-10-years-later.pdf?force_download=true.